Minting a Coin

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Intro

This simple to state puzzle is about flipping coins. The twist is that the probability with which the coin lands on heads is also a random variable.

There is a solution to this riddle which involves computation and integrals, but given the solution, you may suspect there is a clever way to do it. And indeed there is.

Riddle

I am minting a random coin. First I pick a value p uniformly at random from the interval $[0, 1]$. Then I mint a coin such that the probability that the coin lands on head is p.

I sell this random coin to Bob for a pretty penny, and Bob flips it 100 times.

Easy Question: What is the expected number of heads?

Riddle Question: Consider the probability that we get k heads. For what value(s) of k is this probability maximized?

Go to the next page to see the answer, or small hints.

Answer

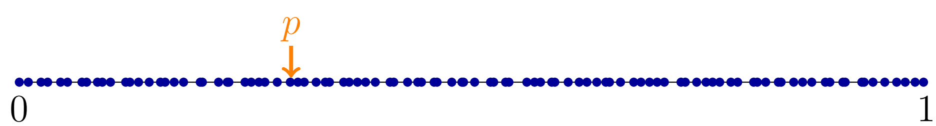
After some thought, you may realize that the experiment is completely symmetric between heads and tails. Hence the expected number of heads after 100 flips is 50.

Hint: Consider how one might simulate this experiment on computer. Clearly you have to generate p uniformly at random between 0 and 1. Then you have to simulate 100 Bernoulli coin flips with success probability p. Can I run this experiment by simply sampling 101 independent values uniformly from $[0, 1]$?

Answer: Indeed, you can simulate this experiment by generating 101 values $\{x_0, x_1, x_2, \ldots, x_{99}, x_{100}\}$, each independently from a uniform distribution on $[0, 1]$. Then the number of heads is exactly the number of x values which fall below $p = x_0$.

Here's the kicker. Instead of forcing the first uniform value to be p, I will *first* generate 101 random values, and then pick one at random to represent the heads probability of my coin. Notice that this way to run the experiment is completely equivalent! That is, the probability that there are k heads is unchanged.

Here is a plot of a possible outcome of the 101 generated values, along with a choice of which one represents p:



Notice that if p is chosen to be the kth smallest value, then this outcome represents $k - 1$ heads. This means that the probability of k heads is exactly $\frac{1}{101}$ for all k.

What have we shown? The probability of flipping k heads is independent of k! (Not a factorial!)

In other words, all outcomes are equally likely. $\square$